**Goal**: Primes in arithmetic progressions

**Recall**: $\pi(x; q, a) := \sum_{\substack{p \le x \\ p \equiv a (q)}} 1$

$$\theta(x; q, a) := \sum_{\substack{p \le x \\ p \equiv a(q)}} \log p \,, \quad \psi(x; q, a) = \sum_{\substack{n \le x \\ n \equiv a(q)}} \Lambda(n)$$

For a character $\chi \ (\mathrm{mod}\ q)$, define

$$\psi(x, \chi) := \sum_{n \le x} \chi(n) \Lambda(n)$$

Orthogonality of characters:

$$\psi(x; q, a) = \frac{1}{\varphi(q)} \sum_{\chi (\mathrm{mod}\ q)} \overline{\chi(a)} \psi(x, \chi)$$

For $\mathrm{Re}(s) > 1$, $\displaystyle \sum_{n \ge 1} \frac{\chi(n) \Lambda(n)}{n^s} = -\frac{L'(s, \chi)}{L(s, \chi)}$

- (partial fraction expansion for $L(s, \chi)$)

Let $\chi$ primitive character modulo $q$ (for $q \ge 2$)

Then for $-1 \le \sigma \le 2$, $s = \sigma + it$

$$\frac{L'}{L}(\sigma + it) = \sum_{\rho : |s - \rho| \le 1} \frac{1}{s - \rho} + O(\log(q(|t| + 1))).$$

(the sum over non-trivial zeros $\rho$ of $L(s,\chi)$ with multiplicity)

<u>Theorem</u>: There exists an absolute constant $c>0$ such that if $\chi$ a character mod $q$, then the region $R_q = \left\{ s = \sigma + it : \sigma > 1 - \frac{c}{\log(q(|t|+1))} \right\}$ contains no zero of $L(s,\chi)$, unless $\chi$ is quadratic, in which case $L(s,\chi)$ has at most one, necesarilly real and simple zero in $R_q$.

<u>Proof</u>: WMA $\chi$ is primitive (if $\chi$ is induced by $\chi^*$, the zeros of $L(s,\chi)$ and $L(s,\chi^*)$ coincide on $Re(s)>0$).

For $\sigma>1$,

$$Re\left[ -3\frac{L'}{L}(\sigma,\chi_0) - 4\frac{L'}{L}(\sigma+it,\chi) - \frac{L'}{L}(\sigma+2it,\chi^2) \right] \geq 0.$$

$$\frac{L'}{L}(\sigma,\chi_0) = \sum_{(n,q)=1} \frac{\Lambda(n)}{n^\sigma} \leq -\frac{\zeta'}{\zeta}(\sigma) = \frac{1}{\sigma-1} + O(1).$$

Let $\rho = \beta + it$ be a zero of $L(s,\chi)$, $\beta > 1 - \frac{\delta}{\log(q(|t|+1))}$.

From partial fraction expansion, $s=\sigma+it$

$$\operatorname{Re}\left(-\frac{L'}{L}(\sigma+it,\chi)\right)\le-\operatorname{Re}\left(\frac{1}{s-\beta}\right)+O(\log(q(|t|+2))$$
$$=-\frac{1}{\sigma-\beta}+O(\log(q(|t|+2))).$$

<u>Case 1</u>: $\chi^2$ ~~complex~~, $\chi^2\neq\chi_0$ ✓ (last time)

<u>Case 2</u>: $\chi$ quadratic, $\chi^2=\chi_0$,

For $\operatorname{Re}(s)>1$,

$$\frac{L'(s,\chi_0)}{L(s,\chi_0)}-\frac{\zeta'}{\zeta}(s)=\sum_{p|q}\frac{p^{-s}\log p}{1-p^{-s}}\ll\sum_{p|q}\log p\ll\log q$$

(Since $L(s,\chi_0)=\zeta(s)\prod_{p|q}\left(1-\frac{1}{p^s}\right)$)

$$\Rightarrow\operatorname{Re}\left(-\frac{L'}{L}(\sigma+2it,\chi^2)\right)=-\operatorname{Re}\left(\frac{\zeta'}{\zeta}(\sigma+2it)\right)+O(\log q)$$
$$\le\operatorname{Re}\left(\frac{1}{\sigma+2it-1}\right)+O(\log(q(|t|+2))$$

(Using partial fraction expansion of $\zeta(s)$).

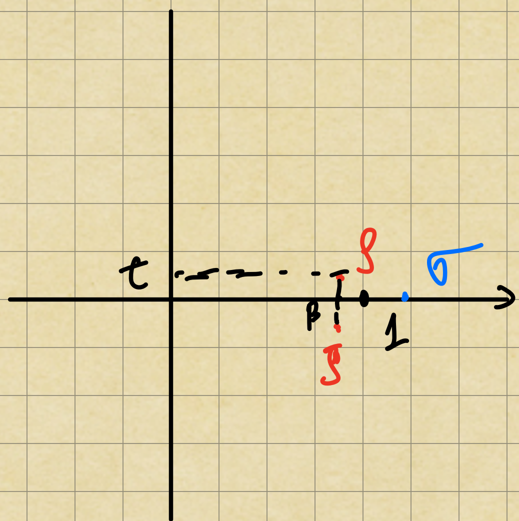
<u>Case 2.1</u>: $|t|\ge\frac{\delta}{\log q}$. Choose $\sigma=1+\frac{4\delta}{\log(q(|t|+2))}$

Then $\operatorname{Re}\left(\frac{1}{\sigma-1+2it}\right)=\frac{\sigma-1}{(\sigma-1)^2+(4t)^2}\ll\log q.$

So we have $\frac{4}{\sigma-\beta} \le \frac{3}{\sigma-1} + O(\log(q(|t|+2)))$.

Obtain same contradiction as before ✓

<u>Case 2.2</u>: $0 < |t| < \frac{\delta}{\log q}$.




Since $\chi$ is real ($\chi = \bar{\chi}$), if $\rho$ is a zero of $L(s,\chi)$, $\bar{\rho}$ is also a zero.

Here

$$-\frac{L'}{L}(\sigma,\chi) \le -\frac{1}{\sigma-\beta-it} - \frac{1}{\sigma-\beta+it} + O(\log q)$$

$$= -\frac{2(\sigma-\beta)}{(\sigma-\beta)^2+t^2} + O(\log q). \qquad (*)$$

On the other hand,

$$-\frac{L'}{L}(\sigma,\chi) = \sum_n \frac{\Lambda(n)\chi(n)}{n^\sigma} \ge -\sum_n \frac{\Lambda(n)}{n^\sigma} = \frac{\zeta'}{\zeta}(\sigma)$$

$$\ge -\frac{1}{\sigma-1} + O(1). \qquad (**)$$

Choose $\sigma = 1 + \frac{2\delta}{\log q}$. Then $|t| \le \frac{1}{2}(\sigma-1) \le \frac{1}{2}(\sigma-\beta)$

So $\dfrac{2(\sigma-\beta)}{(\sigma-\beta)^2+t^2} \geq \dfrac{2(\sigma-\beta)}{(\sigma-\beta)^2+\frac{1}{4}(\sigma-\beta)^2} = \dfrac{8}{5(\sigma-\beta)}$.

$\circledast, \circledast$

$$\Longrightarrow -\frac{\log q}{2\delta} \leq -\frac{8/5}{(\sigma-\beta)} + O(\log q)$$

(Recall $\beta > 1 - \dfrac{\delta}{\log(q(|t|+2))} \geq 1 - \dfrac{\delta}{\log q}$).

so $\sigma - \beta < \dfrac{3\delta}{\log q}$

$$\Longrightarrow -\frac{1}{2\delta} \leq \frac{-8/5}{3\delta} + O(1)$$

Contradiction if $\delta$ small enough. ✓

Therefore $t=0$, so $\rho$ must be real.

<u>Uniqueness</u>: Assume there are two zeros

$$1 - \frac{\delta}{\log q} \leq \beta_1 \leq \beta_2 \leq 1 \quad (\text{possibly equal}).$$

Same argument shows $-\dfrac{1}{\sigma-1} \leq -\dfrac{2\sigma-\beta_1-\beta_2}{(\sigma-\beta_1)(\sigma-\beta_2)} + O(\log q)$

Choose $\sigma = 1 + \dfrac{4\delta}{\log q}$, we obtain the contradiction

$$-\frac{\log q}{4\delta} \le -\frac{8}{25}\frac{\log q}{\delta} + O(\log q).$$
$\square$

<u>Lemma (Landau)</u>: Let $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$ two distinct, real, primitive characters. Let $\beta_j$ a real zero of $L(s,\chi_j)$. Then $\min(\beta_1,\beta_2) < 1 - \frac{c}{\log q_1 q_2}$,

for some (universal) $c > 0$.

<u>Proof</u>: For $\sigma > 1$, we have

$$-\frac{\zeta'}{\zeta}(\sigma) - \frac{L'}{L}(\sigma,\chi_1) - \frac{L'}{L}(\sigma,\chi_2) - \frac{L'}{L}(\sigma,\chi_1\chi_2)$$
$$= \sum_n \frac{\Lambda(n)}{n^\sigma}\left(1+\chi_1(n)+\chi_2(n)+\chi_1\chi_2(n)\right)$$
$$= \sum_n \frac{\Lambda(n)}{n^\sigma}(1+\chi_1(n))(1+\chi_2(n)) \ge 0.$$

Note that $\chi_1\chi_2$ is NOT the principal character modulo $q_1 q_2$ (otherwise $\chi_1$ and $\chi_2$ would induce some character modulo $q_1 q_2$).

So $-\frac{\zeta'}{\zeta}(\sigma) = \frac{1}{\sigma-1} + O(1)$

$$-\frac{L'}{L}(\sigma,\chi_i) \le -\frac{1}{\sigma-\beta_i} + O(\log q_i), \quad i=1,2$$

$$-\frac{L'}{L}(\sigma, \chi_1\chi_2) \le O(\log(q_1 q_2))$$

So
$$\frac{1}{\sigma-\beta_1} + \frac{1}{\sigma-\beta_2} \le \frac{1}{\sigma-1} + O(\log q_1 q_2).$$

Choose
$$\sigma = 1 + \frac{\delta}{\log(q_1 q_2)}$$

$$\Rightarrow \frac{2}{\sigma - \min(\beta_1,\beta_2)} \le \frac{\log q_1 q_2}{\delta} + O(\log q_1 q_2)$$

$$\Rightarrow \min(\beta_1,\beta_2) \le 1 - \frac{1}{\log(q_1q_2)}\left(\frac{2}{\delta^{-1}+O(1)} - \delta\right). \square$$

<u>Corollary</u>: There is at most one character modulo $q$ with a Siegel zero (a real zero with $\beta > 1 - \frac{c}{\log q}$).

Moreover, for $Q \ge 3$, there is at most one $q \le Q$ for which it exists a primitive character $\chi$ mod $q$ with a real zero $\beta > 1 - \frac{c}{\log Q}$.

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Using this, we obtain:

<u>Theorem</u>: There exists a constant $c>0$ such that if $q \le \exp(2C\sqrt{\log x})$ and $L(s,\chi)$ has no exceptional zero, then

$$\psi(x,\chi) = 1_{\chi=\chi_0} x + O(x\exp(-c\sqrt{\log x})).$$

If $L(s,\chi)$ has an exceptional zero $\beta$, then

$$\psi(x,\chi) = -\frac{x^\beta}{\beta} + O(x\exp(-c\sqrt{\log x})).$$

<u>Proof</u>: Exercise.

From orthogonality of characters:
If no character mod $q$ has Siegel zero, then for $q \le \exp(2C\sqrt{\log x})$,

$$\psi(x;a,q) = \frac{1}{\varphi(q)}x + O(x\exp(-c\sqrt{\log x})).$$

If there exists a character $\chi$ (mod $q$) with a Siegel zero $\beta$, then

$$\psi(x;a,q) = \frac{1}{\varphi(q)}x - \frac{1}{\varphi(q)}\overline{\chi}(a)\frac{x^\beta}{\beta} + O(x\exp(-c\sqrt{\log x})).$$

Beyond scope of this course: it is possible to obtain some (ineffective) upper bounds on $\beta$.

# Alternative proof of PNT (Sheet 12, ex 2)

We use zero-free region of $\zeta(s)$:

There exists a constant $C>0$ s.t. if $\rho = \sigma + it$ is a non-trivial zero of $\zeta(s)$, then

$$\sigma > 1 - \frac{C}{\log(2+|t|)}.$$

From truncated Perron, for $c = 1 + \frac{1}{\log x}$,

$$\psi(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \left(-\frac{\zeta'}{\zeta}(s)\right) x^s \frac{ds}{s} + O\left(\frac{x(\log x)^2}{T}\right).$$

(for $T \le x$)

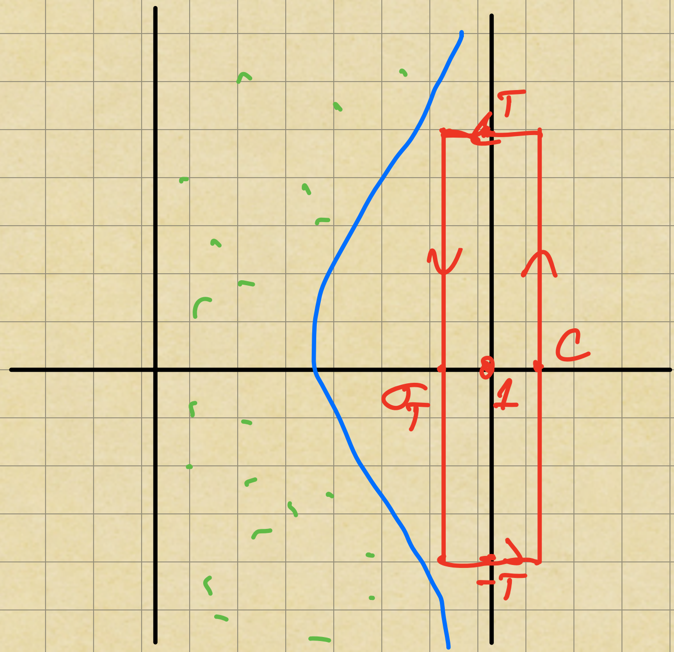
Let $F(s) = -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}$, integrate along box with corners $c-iT, c+iT, \sigma_T+iT, \sigma_T-iT$, with $\sigma_T = 1 - \frac{C}{2\log T}$.

$\zeta(s)$ has no zeros in this region, so there is only the pole at $s=1$ for $F(s)$ in the box.

Hence

$$\frac{1}{2\pi i}\left(\int_{c-iT}^{c+iT} F(s)\,ds + \int_{c+iT}^{\sigma+iT} F(s)\,ds + \int_{\sigma+iT}^{\sigma-iT} F(s)\,ds + \int_{\sigma-iT}^{c-iT} F(s)\,ds\right)$$

$$= \operatorname{Res}_{s=1} F(s) = x.$$




We have that if $s=\sigma+it$ with $|t|\geq 2$ and $\sigma_T \leq \sigma \leq c$, then

$$-\frac{\zeta'}{\zeta}(\sigma+it) \ll \log(2+|t|)^2$$

( because $|s-\rho| \gg \frac{1}{\log(2+|t|)}$ for any $\rho$ zero of $\zeta(s)$ )

Second integral bounded by

$$\int_{c+iT}^{\sigma+iT}\left(-\frac{\zeta'}{\zeta}(s)\right)\cdot\frac{X^s}{s}\,ds \ll (c-\sigma)\cdot\frac{X}{T}\cdot(\log T)^2$$

$$\ll \frac{X}{T}(\log T)^2$$

Similar bound for integral IV

Integral III: $\displaystyle\int_{\sigma+iT}^{\sigma-iT}\left(-\frac{\zeta'}{\zeta}(s)\frac{X^s}{s}\right)ds \ll X^{\sigma_T}(\log T)^2\int_0^T\frac{1}{1+t}\,dt$

$$\ll X^{1-\frac{c}{\log T}}\cdot(\log T)^3.$$

Hence $\psi(x) = x + O\left(\frac{x}{T}(\lg x)^2 + x^{1-\frac{c}{2\lg T}} \cdot (\lg T)^3\right)$.

Choose $T = \exp(\sqrt{\lg x})$.

✓